UDC 539. 3

ON A CASE OF THE INVERSE PROBLEM OF TWO-DIMENSIONAL THEORY OF ELASTICITY

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The problem of determining the contours of a finite number of holes of equal strength in a statically loaded plane under the condition that the given normal stresses on the outline of each hole take on constant but distinct values is considered. The problem is converted by a method analogous to that used for the case of an identical load on the whole domain boundary [1], into a regular integral equation which is solved effectively on an electronic computer, while a closed solution is obtained for certain cases. A plane with one hole under an arbitrary load is also considered. Numerical examples are given. An assertion formulated in [2] about the property of greatest strength of contours of equal strength is proved.

Let S denote the plane of the complex variable z under consideration which has n holes. Let the function $\omega_0(\zeta) = C\zeta + \omega(\zeta)$ map the canonical domain F of the variable ζ conformally on S with the infinitely remote points in correspondence; $\omega(\zeta)$ is holomorphic in F and bounded at infinity. A plane with n circular holes is selected as F (with n parallel slits in [2]). The boundary conditions of the first boundary value problem in the transformed domain become

$$\sigma_r + \sigma_\theta = 4\,\mathrm{Re}\,(\Phi(\xi)) \tag{1}$$

$$\sigma_\theta - \sigma_r + 2i\tau_{r\theta} = \frac{2(\xi - a_k)^2}{r_k^2\,\overline{\omega_0'(\xi)}}\left(\overline{\omega_0(\xi)}\,\Phi'(\xi) + \omega_0'(\xi)\,\Psi(\xi)\right); \tag{2}$$

$$\xi \in L_k, \quad k = 1, \ldots, n$$

where σ_r, σ_θ, $\tau_{r\theta}$ are stress components in a polar coordinate system with pole at the point a_k, which is the center of the circle L_k of radius r_k; σ_r and $\tau_{r\theta}$ are given. The functions $\Phi(\zeta)$ and $\Psi(\zeta)$ which are holomorphic in F have the following asymptotic at infinity

$$\Phi(\zeta) = b + O(\zeta^{-2}), \quad b = {}^1/_4\,(\sigma_x^\infty + \sigma_y^\infty)$$

$$\Psi(\zeta) = a + O(\zeta^{-2}), \quad a = {}^1/_2\,(\sigma_y^\infty - \sigma_x^\infty) + i\tau^\infty$$

(σ_x^∞, σ_y^∞, τ^∞ is a given uniform stress field).

The following theorem is formulated without proof in [2]: for $\sigma_r = p =$ const the stress $\sigma_\theta = 4b - p$ is a minimum on the contours of equal strength as compared with the maximum quantity σ_θ on any other hole contours.

The proof results from the following assertion.

If a real function $u(\zeta)$ which is not a constant and is harmonic in a plane with a finite number of arbitrary smooth holes, tends to a definite limit A at infinity, then the inequality

$$\min(u(\xi)) < A < \max(u(\xi)), \quad \xi \in L \tag{3}$$

is valid on the hole boundary L.

Indeed, let the left side of (3), say, not be satisfied, i.e. $u(\zeta) > A$ for all $\xi \in L$. Then by the property of the extremum of harmonic functions $u(\zeta) > A$ at all points of the domain, in particular, on a circle γ of sufficiently large radius R which contains all the holes. By the theorem of the mean applied to $u(\zeta)$ in the exterior of γ, we have

$$A = \frac{1}{2\pi R}\int_{\gamma} u(\zeta)\, dS$$

In combination with the inequality $u(\xi) > A$ we obtain $u(\zeta) = A$ on γ from which it follows that $u(\zeta) = A$ outside γ, meaning everywhere in the domain, which is a contradiction. Analogously for the right side in (3). Applying the assertion proved to the function $\operatorname{Re}\Phi(\zeta)$, we obtain from (1) for $\sigma_r = p$

$$\min(\sigma_\theta(\xi)) + p < 4b < \max(\sigma_\theta(\xi)) + p, \quad \xi \in L$$

from which it follows that either $\max|\sigma_\theta(\xi)| > \sigma_x^\infty + \sigma_y^\infty - p$ o $\Phi(\zeta)$ is constant in F In this latter case the contours are of equal strength $\sigma_\theta(\xi) = \sigma_x^\infty + \sigma_y^\infty - p$ on L.

The property mentioned of the greatest strength of equal strength contours holds for $n = 1$ even in the case of an arbitrary static load $\sigma_r = \sigma_r(\xi)$, $\tau_{r\theta} = \tau_{r\theta}(\xi)$, as follows from the relationship (1), integrated with respect to θ

$$\int_0^{2\pi}(\sigma_\theta + \sigma_r)\, d\theta = 2\pi(\sigma_x^\infty + \sigma_y^\infty)$$

Hence we have

$$\max|\sigma_\theta(\theta)| \geqslant \sigma_x^\infty + \sigma_y^\infty - \langle\sigma_r\rangle$$

Equality is achieved at each point only on a contour of equal strength.

To find the contour shape, let us set $a_1 = 0$, $r_1 = 1$ and let us rewrite (2) in the form

$$\overline{(\omega_0(\xi)\,\Phi(\xi) + \omega_0'(\xi)\,\Psi(\xi))} = \overline{\xi^2\omega_0'(\xi)}\,(\sigma_\theta - \sigma_r + 2i\tau_{r\theta}) \tag{4}$$

Following [3], we use the boundary condition (4) on a circle to solve the direct problem, the determination of $\sigma_\rho(\xi)$ by means of a given $\omega_0(\zeta)$ by reducing it to a degenerate singular equation, and then to an infinite algebraic system

$$a_{m+2} - \sum_{k=0}^{m}(m-k+1)\overline{C}_{m-k+1}a_k - (m+1)\sum_{k=1}^{\infty}\overline{C}_{m+k+1}\overline{a}_k = A_m,$$
$$m = 0, 1, 2, \ldots$$

where $\{A\}$ and $\{C\}$ are known but $\{a\}$ are unknown coefficients of the expansions

$$\sigma_r + \sigma_\theta = \sum_{-\infty}^{\infty} a_k\xi^k, \quad a_{-k} = \bar{a}_k \tag{5}$$

$$\omega_0(\zeta) = C\zeta + \sum_{k=1}^{\infty} C_k\zeta^{-k} \tag{6}$$

$$-\xi^2\omega_0'(\xi)(\sigma_r + i\tau_{r\theta}) = \sum_{-\infty}^{\infty} A_k\xi^k \tag{7}$$

If the coefficients interchange roles, we obtain an algebraic system to determine $\omega(\zeta)$ for an appropriate renumbering.

Let us examine an example. Let the load $\sigma_x^\infty = \sigma_y^\infty = \tau^\infty = \tau_{r\theta} = 0$, $\sigma_r = 1 - \cos 2\theta$, be given, then $\sigma_\theta = -1$ on the contour of equal strength. Hence $a_2 = a_{-2} = -1$ in their expansion (5). The remaining a_k are zero and the matrix of the system becomes tridiagonal

$$4C_1 - 5C_3 = -1$$
$$(2k-1)C_{2k-1} - 4(2k+1)C_{2k+1} + (2k+5)C_{2k+3} = 0$$

The system was solved numerically. The order was chosen equal to 1000. The shape of the contour of equal strength (a) and the shape of the load (b) are presented in Fig. 1.

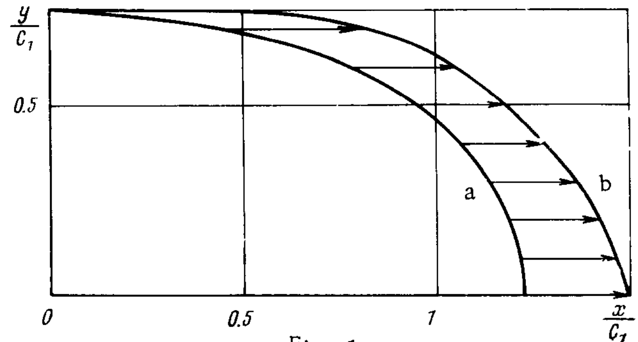

Fig. 1

Let us turn to a multiply connected domain. Considering (1) in the case of a load variable in ξ as a Dirichlet problem with respect to the real part of the function $\Phi(\zeta) - 4b$ holomorphic in F which decreases at infinity, we conclude that for given σ_r and b the stresses $\sigma_\theta(\xi)$ should satisfy known orthogonality conditions (4). In particular, the problem is solvable if $\sigma_\theta(\xi)$ takes on constant but generally different values on each contour L_k (modified Dirichlet problem [4]). We shall also call the appropriate contours of equal strength. Whether they exist or not depends on the solvability of the one-sided boundary value problems (2) with the condition contaning the derivatives

$$\omega_0'(\xi)\Psi(\xi) + \overline{\omega_0(\xi)}\Phi'(\xi) + \frac{\partial\bar{\xi}}{\partial\xi}\lambda_k(\xi)\overline{\omega_0'(\xi)} = 0 \qquad (8)$$
$$\frac{\partial\bar{\xi}}{\partial\xi} = -\frac{r_k^2}{(\xi - a_k)^2}, \quad \xi \in L_k$$
$$\lambda_k = {}^1/_2\,(\sigma_\theta - \sigma_r + 2i\tau_{r\theta})$$

Condition (8) simplifies in a case of practical importance and the problem is solved completely. Namely, let

$$\sigma_r(\xi) = p_k, \quad \tau_{r\theta}(\xi) = 0, \quad \xi \in L_k, \quad k = 1, 2, \ldots, n$$

Then $\sigma_\theta(\xi) = \sigma_x^\infty + \sigma_y^\infty - p_k$ should hold on the contours of equal strength, hence $\Phi'(\zeta) = 0$ in F and (8) becomes

$$H'(\xi) + \lambda_k \frac{\partial\bar{\xi}}{\partial\xi}\overline{\omega'(\xi)} = C\lambda_k \frac{r_k^2}{(\xi - a_k)^2} - Ca \qquad (9)$$

$$H'(\zeta) = \omega'(\zeta)(\Psi(\zeta) - a), \quad \lambda_k = {}^1/_2\,(\sigma_\theta - p_k)$$

Let us assume that all the λ_k are not zero; $H'(\zeta)$ and $\omega'(\zeta)$ belong to a class P of functions holomorphic in F and continuous to L. If the additional condition that they are derivatives of functions from P is discarded, then (9) is a particular case of the problem studied [5] for a finite $(n+1)$ -connected domain: Find the pair of functions $f(\zeta)$ and $g(\zeta)$ from P by means of the boundary condition

$$f(\xi) + v(\xi)\overline{g(\xi)} = h(\xi) \tag{10}$$

where $v(\xi)$ and $h(\xi)$ given on L are Hölder continuous, and $v(\xi)$ does not vanish anywhere. First establishing that every solution (10) from P is also Hölder continuous on L under these conditions, the author of (5) reduces (10) to an equivalent system of singular integral equations of normal type by representing $f(\zeta)$ and $g(\zeta)$ by Cauchy type integrals with real densities.

The adjoint system hence turns out to be related to the conjugate problem (s is the arclength of the contour)

$$f_1(\xi) + v^{-1}(\xi)\left(\frac{\partial\bar{\xi}}{\partial s}\right)^2\overline{g_1(\xi)} = 0 \tag{11}$$

The difference between the numbers l and l_1 of the solutions of the homogeneous problems (10) and (11), which are linearly independent over the field of real numbers, equals $2m - 2(n-1)$, where m is the index of the function $v(\xi)$ on L. For a plane with n holes, $(n-1)$ should be replaced by n, if a decrease at infinity is required of $f(\zeta)$ and $g(\zeta)$

According to what has been proved $H(\zeta)$ and $\omega'(\zeta)$ in (9) possess first derivatives whose boundary values on L are Hölder continuous. Hence using the I. N. Vekua [4] integral representation for H and ω we obtain that the conjugate problem to (9) has the form

$$p(\xi) + \lambda_k^{-1}\left(\frac{\partial\bar{\xi}}{\partial s}\right)^2\overline{q(\xi)} = 0 \tag{12}$$

where l' and l_1' are the numbers of linearly independent solutions of the homogeneous problems (9) and (12) connected by the relationship $l_1' - l' = 2n$. The identity and condition

$$\left(\frac{\partial\bar{\xi}}{\partial\xi}\right) = \left(\frac{\partial\bar{\xi}}{\partial s}\right)^2, \quad \xi \in L; \quad \lambda_k \neq 0, \quad \operatorname{Im}\lambda_k = 0 \tag{13}$$

were used in deducing (12).

When $v(\xi) = \lambda_k$ on L_k the problems (11) and (12) coincide, hence, $l_1' = l_1$, meaning $l' = l$, but the homogeneous problem (10) has only a zero solution. In fact, it follows from (1) that $\operatorname{Re}(if(\xi)\,g(\xi)) = 0$ on L from which $f(\zeta) = g(\zeta) = 0$ in F. Therefore, l' is also zero.

Let us integrate (9) with respect to

$$H(\xi) + \lambda_k\overline{\omega(\xi)} - d_k = -\frac{Cr_k^2\lambda_k}{\xi - a_k} - Ca\xi \tag{14}$$

(d_k are constants of integration). For a zero right side, the problem (14) has only a zero solution in the class of holomorphic functions together with Hölder-continuous first derivatives on L, as has been proved; hence, all the $d_k = 0$.

To seek $\omega(\zeta)$ in fact, let us assume by using the D. I. Sherman method [6]

$$H(\zeta) = \frac{1}{2\pi i}\int\limits_L \frac{u(\xi)}{\zeta - \xi}\,d\xi, \quad d_k = \int\limits_{L_k} u(\xi)\,dS \tag{15}$$

$$\omega(\zeta) = \frac{1}{2\pi i}\sum_{k=1}^{n}\int\limits_{L_k} \frac{\overline{u(\xi)}}{\lambda_k(\zeta - \xi)}\,d\bar{\xi}, \quad dS = |d\xi|$$

The functions $H(\zeta)$ and $\omega(\zeta)$ will possess the required boundary properties if it is assumed that the function $u'(\xi)$ given on L is continuous and $u(\xi)$ is integrable[4]. Substituting (15) into (14) and using Sokhotskii formula, we obtain

$$u(\xi_0) + \frac{1}{2\pi i}\int\limits_L u(\xi)\,d\ln\frac{\xi - \xi_0}{(\bar{\xi} - \bar{\xi}_0)^{\alpha_{jk}}} + \int\limits_{L_k} u(\xi)\,dS = \tag{16}$$

$$C\left(\frac{r_k\,\Lambda_k}{\xi_0 - a_k} + a\xi_0\right); \quad \alpha_{jk} = \lambda_j\lambda_k^{-1}, \quad \xi \in L_j, \quad \xi \in L_k$$

Equation (16) is regular by construction [6]. It is always solvable uniquely i.e., the corresponding homogeneous equation is not solvable.

In fact, let $u_0(\xi)$ denote the solution of the homogeneous equation. The functions $H_0(\zeta)$ and $\omega_0(\zeta)$ constructed by using (15) and the solution will satisfy the boundary condition (10) with zero right side, and therefore, are zero in F. There hence results that $u_0(\xi)$ and $\overline{u_0(\xi)}/\lambda_k$, meaning, $\overline{u_0(\xi)}$ also, are boundary values of functions holomorphic in n simply-connected domains bounded by circles L_k, hence $u_0(\xi)$ reduces to a constant on each contour. Finally, using the equality $d_k^\circ = 0$ we find that these constants are also zero.

The solution (16) by least squares, by means of the correspondingly compli - cated formulas in [1], was carried out numerically. The solid lines in Fig. 2 show the system of contours of equal strengths for the following data:

$$n = 4, \quad a_1 = -a_3 = 1.1, \quad a_2 = -a_4 = i$$
$$r_k = 0.7, \quad k = 1, \dots, 4$$
$$\sigma_x^\infty = \sigma_y^\infty = \tau^\infty = 0, \quad \tau_{r\theta} = 0$$
$$p_1 = p_3 = -1, \quad p_2 = p_4 = 1$$

If the geometric properties of the domain S are such that it is mapped uni-valently on a plane with slits along one line, then the problem sometimes admits of solution by quadratures.

Let us consider an example. Let a plane with three holes be mapped on a plane with slits along the real axis $(-\nu_2, -\nu_1)$, $(-1,1)$ and (ν_1, ν_2), $1 < \nu_1 < \nu_2$ (symmetric case). Let l_0 denote the middle slit and l_1, l_2 the outer slits. We assume that the only nonzero component on l_0 is $\sigma_r = 1$ while $\sigma_r = -1$ on $l_{1,2}$. Then $\sigma_\theta = -\sigma_r$, $\lambda_0 = -1$, $\lambda_{1,2} = 1$. The functions $\omega'(\zeta)$ and $H'(\zeta)$ have power-law singu - larities in the order of $1/2$ at the ends of the slits, and are bounded in the rest of the plane [2]. On the real axis $\partial\bar{\xi}/\partial\xi = 1$. By separating real and imaginary parts the boundary value problem (9) is reduced to two mixed problems of the theory of holo - morphic functions [4] (the variable ξ is real)

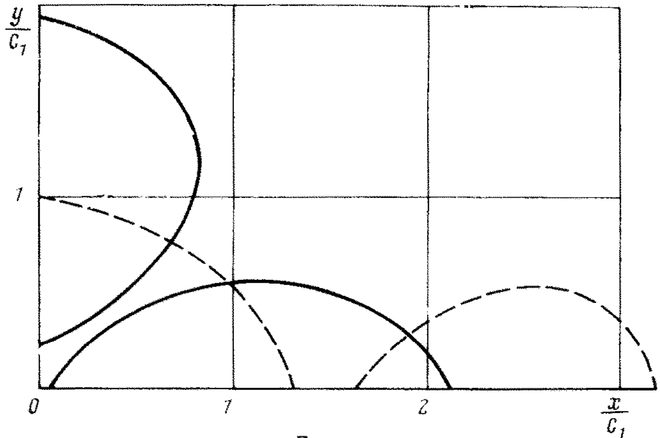

Fig. 2

$$\operatorname{Re} P(\xi) = 0, \quad \xi \in l_0; \quad \operatorname{Im} P(\xi) = 0, \quad |\xi| > 1$$

$$P(\zeta) = H'(\zeta) + \omega'(\zeta)$$

$$\operatorname{Re} Q(\xi) = 0, \quad \nu_1 < |\xi| < \nu_2$$

$$\operatorname{Im} Q(\xi) = 0, \quad |\xi| > \nu_2, \quad |\xi| < \nu_1$$

$$Q(\zeta) = H'(\zeta) - \omega'(\zeta)$$

whose solution in the mentioned class is found by the formulas of M. V. Keldysh-L. I. Sedov [4] with the asymptotics taken into account

$$P(\zeta) = \frac{Ca\zeta + d_1}{\sqrt{\zeta^2 - 1}}, \quad Q(\zeta) = \frac{Ca\zeta + d_2}{\sqrt{(\zeta^2 - \nu_1^2)(\zeta^2 - \nu_2^2)}}$$

Integrating, we obtain the equation of the contours in Cartesian coordinates [7] (for $x > 0$ and $y > 0$).

For the central contour

$$x = \frac{C_1}{2a}\left[\left(a\nu_2 - \frac{d_2}{\nu_2}\right) F(\varphi, k) - a\nu_2 E(\varphi, k)\right]$$

$$\varphi = \arcsin\frac{\xi}{\nu_1}, \quad k = \frac{\nu_1}{\nu_2}, \quad y = \frac{C_1}{2}\sqrt{1 - \xi^2}, \quad d_1 = 0$$

For the right side contour

$$x_0 = x_0 + \frac{C_1}{2}\sqrt{\xi^2 - 1}$$

$$y = \frac{C_1}{2a}\left[a\nu_2\left(\mathbf{E}(k_1) - E(\varphi, k_1)\right) - \frac{d_2}{\nu_2}\left(\mathbf{K}(k_1) - F(\varphi, k_1)\right)\right]$$

$$\varphi = \arcsin\sqrt{\frac{\nu_2^2 - \xi^2}{\nu_2^2 - \nu_1^2}}, \quad k_1 = \sqrt{1 - \frac{\nu_1^2}{\nu_2^2}}$$

$$x_0 = \frac{C_1}{2}\left[\sqrt{\nu_2^2 - 1} + \left(\nu_2 - \frac{d_2}{c\nu_2}\right)\mathbf{K}(k) - \nu_2\mathbf{E}(k)\right]$$

Here F, E are elliptic integrals ($\mathbf{K}$, $\mathbf{E}$ are the complete elliptic integrals) of the first and second kinds.

By virtue of symmetry, d_2 is determined from the condition : $y = 0$ for $\xi = \nu_2$. We obtain

$$d_2 = \nu_2^2 \mathbf{E}(k_1) / \mathbf{K}(k_1)$$

Analogous computations for two holes under a constant load are performed in [2]. The dashed lines in Fig. 2 present the contour shapes for $\nu_1 = 1.1$, $\nu_2 = 2.1$.

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